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On Intertemporal Preferences for Uncertain Consumption: A Continuous Time Approach

by
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Stanford University
and
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Massachusetts Institute of Technology

WP# 2605-87-EFA Revised May 1991

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On Intertemporal Preferences for Uncertain Consumption: A Continuous Time Approach[*]

Ayman Hindy[†] and Chi-fu Huang[‡]

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Abstract

We propose a family of topologies on the space of consumption patterns in continuous time under uncertainty. Preferences continuous in any of the proposed topologies treat consumptions at nearby dates as almost perfect substitutes except possibly at information surprises. The topological duals of the family of proposed topologies essentially contain processes that are the sums of processes of absolutely continuous paths and martingales. Thus if equilibrium prices for consumption come from the duals, consumptions at nearly adjacent dates in a state of nature have almost equal prices except possibly at information surprises. In particular, if the information structure is generated by a Brownian motion, the duals are composed of Itô processes. We investigate some implications of our topologies on standard models of choice in continuous time as well as on recent models of non time-separable representations of preferences. We also discuss the properties of prices of long-lived assets in economies populated with agents whose preferences are continuous in our topologies when there are no arbitrage opportunities.

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1 Introduction and Summary

Three factors determine the pattern of current prices for a consumption good delivered at different dates in different states of nature. These are: the variations in future supplies as determined by endowment and technology, the variations in demand for future consumption as determined by preferences for consumption at different times and in different states and, finally, the evolution of information about future events. We focus in this paper on intertemporal preferences with two objectives in mind. First, we formalize the notion that a good consumed at one time in one state of nature is "close" to the same good consumed at a nearby time in the same state, except possibly when there is a significant change of information. Put differently, consumptions at nearby times in one state are almost perfect substitutes except possibly at information surprises. Second, we show that, with preferences which exhibit this notion, prices for consumption at two adjacent dates in one state are very close except possibly at information surprises. This is a natural result since goods which are almost perfect substitutes should have very close prices.

The intuition we formalize is certainly not new. We all experience effects of recent past consumption on current appetite. In addition, we also share the intuition that in an efficient market asset prices should not suddenly jump without some sudden change of expectations about future events, or, in other words, without a surprise. To explain this intuition, one could argue as follows. If one knew for sure that the price of a certain asset in a few moments will be much higher than it is now then one would be willing to delay his consumption for these few moments and use the proceeds to buy such an asset. Alternatively, one could borrow from someone else who is willing to delay his consumption, pay him a rate of interest and use the funds to realize the gain in the price of the asset. This behavior would force the price now to increase and become very close to the price of the asset a few moments from now.

Willingness to delay consumption hinges on the perception that consumptions at nearby dates are rather close substitutes. If preferences do not conform to this notion in the sense that no one is willing to delay consumption for a short while for a small fee, then a situation of a big jump in prices over short periods in the absence of any new information might prevail in equilibrium. In fact, this is a prediction of standard models with time-additive utility functions and most models with non-time-additive utility functions. In these models, continuous changes in prices are obtained mainly by exogenously specifying continuously varying endowments or perfectly malleable technology with continuously varying returns.

This paper is a natural extension of the research reported in Hindy, Huang, and Kreps (1991), which we henceforth abbreviate as HH&K. Among the questions HH&K address are: How might one represent a consumption pattern in continuous time under certainty and what are the appropriate topologies on the space of consumption patterns that capture the idea that consumptions at nearly adjacent dates are almost perfect substitutes? Moreover, what form would the equilibrium prices take when individuals in the economy have preferences continuous in the appropriate topologies?

There are "standard" answers for the questions raised by HH&K: A consumption pattern on a time interval, say $[0,1]$, is represented by a real-valued function $c : [0,1] \rightarrow \Re_+$, where $c(t)$ is the consumption "rate" at time t. In addition, the real-valued function $y : [0,1] \rightarrow \Re_+$, with $y(t) = \bar{y}(t) + \int_0^t \theta(s)c(s)\,ds$, for some function $\bar{y}(t)$ and a smooth "averaging" function θ, represents a summary of past consumption experience at each time t. Two consumption patterns are close if they are close as functions in an L^p norm topology for some $1 \leq p < \infty$. An agent's preferences are represented by $\int_0^1 u(c(t), y(t), t)dt$, a time-additive functional of consumption rates and an index of past consumption, where $u(c, y, t)$ is a "felicity" function at time t. The equilibrium prices come from L^q with $\frac{1}{p} + \frac{1}{q} = 1$, the topological dual of L^p; that is, the price at time zero of a consumption pattern c can be represented as $\int_0^1 c(t)z(t)dt$, where z is from L^q and $z(t)$ is interpreted as the time zero price of one unit of consumption per unit time at time t.

HH&K argue that these standard answers are unsatisfactory for the following reasons. First, in modeling consumption over time, one should allow consumption in gulps as well as at rates – observed consumption behavior such as having meals is in gulps. Second and more important, the L^p topology on the space of consumption rates is so strong along the time dimension that consumptions at nearby dates are perfect nonsubstitutes! As a consequence of the strong topology, the space of equilibrium prices is too rich. It includes, for example, discontinuous functions of time and continuous functions of unbounded variation signifying that equilibrium prices for consumptions at nearby dates are either very different or, even though continuous, fluctuate in a nowhere differentiable manner.[1]

HH&K represent a consumption pattern over time under certainty by an increasing,[2] positive, and right-continuous function on $[0, 1]$. Let x be such a function. Then $x(t)$ denotes the cumulative consumption from time zero to time t. If x is an absolutely continuous function of time, its derivatives exist almost everywhere and can be interpreted as consumption rates. The discontinuities of x are the gulps of consumption. Letting X be the linear span of the space of these consumption patterns, HH&K introduce a family of norm topologies on X so that, among other things, consumptions at nearly adjacent dates are almost perfect substitutes. An example of this family of norm topologies is the L^p topology on cumulative consumption, where the L^p-norm of a consumption pattern x is given by [3] $(\int_0^1 |x(t)|^p dt + |x(1)|^p)^{\frac{1}{p}}$. HH&K also show that preferences continuous in any of the norm topologies can be represented by a time-additive functional of the felicities whose value at any date t depends explicitly on the consumption rate at that time, the functional in the "standard answers", only if the felicity functions are linear in consumption rates! The equilibrium prices basically come from absolutely continuous functions – prices of consumptions at nearby dates are almost equal and change over time in an almost differentiable manner.

The analysis in HH&K is closely related to the literature on commodity differentiation in general equilibrium theory due to Mas-Colell (1975) and Jones (1981, 1983). The central question asked in this literature is whether commodities with similar characteristics have similar prices. Jones (1981) explicitly considered the case where the characteristic of a commodity is just the time when it is consumed. From this literature, we have sufficient conditions for consumptions at nearby dates to have almost equal prices. However, the topology considered by Mas-Colell and Jones, the weak* topology on the space of signed measures on a compact set, is stronger than those suggested by HH&K. As a result, the prices for consumption, although continuous functions of time, can fluctuate in a nowhere differentiable fashion and are not absolutely continuous. This feature is counter-intuitive since we normally don't expect prices to fluctuate in a non differentiable manner under certainty. In addition, there is the unpleasant consequence that continuously compounded interest rates do not, in general, exist.

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[1] Note that one might expect equilibrium prices for consumption to fluctuate widely in an economy under uncertainty due to temporal resolution of uncertainty, however; see Huang (1985a, 1985b).

[2] Throughout this paper we will use weak relations: increasing means nondecreasing, positive means nonnegative, etc. When the relations are strict, we will say, for example, strictly increasing.

[3] We include the last term $x(1)$ in the definition of the norm to distinguish between two consumption patterns that are equal over the interval $[0, 1)$, but differ in that one pattern involves a "gulp" at $t = 1$, which will not be accounted for by the integral $\int_0^1 |x(t)|^p dt$.

The purpose of this paper is to develop topologies on the space of consumption patterns under uncertainty that capture the notion that consumptions at adjacent dates are almost perfect substitutes. A crucial distinction between an economy under certainty and an economy under uncertainty is that in the latter the passage of time also reveals partially the true state of nature, or reveals *information.* In a world of uncertainty, the preferences of an individual over consumption patterns may be affected by the way uncertainty is resolved over time. A world in which any uncertainty is resolved gradually over time, with ample "warning" and "preparation" for new bits of information is certainly different from a world which is "sudden" and "full of surprises". Thus it is unreasonable to require *a priori* that consumptions at nearby dates, which may be random, be almost perfect substitutes unless there are no "surprises" there. Similarly, we would not expect the prices for consumption at different dates in a state of nature to be a continuous function of time except at (random) times of no surprise.

We introduce a family of norm topologies on the linear span of the set of consumption patterns under uncertainty, which is composed of processes with positive, increasing, and right-continuous paths. The topologies are natural generalizations of the HH&K topologies. In particular, they reduce to the HH&K topologies when the true state of nature is revealed at time 0 and they imply that consumptions at nearby dates, where there is no discontinuity of information, are almost perfect substitutes. We then show that preferences continuous in any one of the family of topologies exhibit intuitively appealing economic properties. In addition, we show that *almost all* preferences in the literature that are represented by expected utility functionals are not continuous in the topologies we introduce and thus they don't treat consumptions at nearly adjacent dates as close substitutes.

In general equilibrium theory or in the arbitrage pricing of financial assets, prices for consumption normally come from the topological dual spaces. Thus we also characterize the topological duals of the suggested family of norm topologies and show that they are essentially composed of processes which are sums of processes of absolutely continuous paths and martingales. This is a very natural result. HH&K have shown that in the case of certainty the shadow prices for consumption are essentially absolutely continuous functions. Hence preferences that treat consumption at nearly adjacent dates as almost perfect substitutes will give rise to nearly equal prices for consumption at nearly adjacent dates and these prices vary over time in an almost differentiable fashion; an intuitively attractive conclusion. In the case of uncertainty a new element, the pattern of information flow, affects the sample path properties of equilibrium prices. This effect is captured in the martingale component of the price process. It is known that a martingale can make discontinuous changes only at surprises and can fluctuate in a nowhere differentiable fashion. Thus equilibrium prices for consumption in a state of nature at different dates are a continuous function of time except possibly at surprises and can fluctuate in a nowhere differentiable manner due to temporal resolution of uncertainty. This agrees with our economic intuition.

We also investigate properties of arbitrage-free price processes in a dynamic securities market economy where an arbitrage opportunity is defined using concepts of continuity derived from our family of topologies. It is shown that, between lump-sum ex-dividend dates, price processes for long-lived securities are continuous except possibly at surprises. In particular, if the information structure is generated by a Brownian motion and the cumulative dividend process of a security is an absolutely continuous process, the ex-dividend price process for this security is an Itô process–a hypothesis widely used in the earlier literature on financial markets.

It is worthwhile to point out that earlier models of equilibrium using the standard representation of utility had to rely on exogenous factors in addition to preferences to characterize the sample path properties of equilibrium prices; see Duffie and Zame (1989) and Huang (1987). For example, in the case of a Brownian motion filtration, the price process for a security with absolutely continuous cumulative dividend process will not be an Itô process unless the aggregate endowment process itself is an Itô process. In our setup, however, since consumption at nearly adjacent dates are almost perfect substitutes at times of no surprise, price processes of long-lived securities with absolutely continuous cumulative dividends will be Itô processes independently of the special properties of the aggregate endowment process.

The rest of this paper is organized as follows. Section 2 formulates the stochastic environment under study with a time span $[0,1]$. A consumption pattern $x = \{x(t); t \in [0,1]\}$ is a stochastic process, consistent with the information flow, having positive, increasing, and right-continuous sample paths. The random variable $x(t)$ denotes the cumulative consumption from time 0 to time t. Let $\mathbf{X}_+$ be the space of consumption patterns and $\mathbf{X}$ be the linear span of $\mathbf{X}_+$. We define the commodity space and consumption set to be subsets of $\mathbf{X}$ and $\mathbf{X}_+$, respectively, and introduce a family of topologies. Our construction of the the topologies is motivated by two considerations.

First, an economy under certainty is a special case of an economy under uncertainty where the true state of nature is revealed at time 0. In such event, we would like our topology to agree with the HH&K topology. This necessitates that the norms of HH&K be used path by path (state by state) on a consumption pattern. For the L^p example mentioned above, the path-wise distance between $x \in \mathbf{X}_+$ and 0 is $\left(\int_0^1 |x(\omega,t)|^{p(\omega)}dt + |x(\omega,1)|^{p(\omega)}\right)^{\frac{1}{p(\omega)}}$, where $x(\omega,t)$ is the value of the random variable $x(t)$ when the state of nature is ω. Note that in the above path-wise distance, p is a function of the state of nature ω – there is no *a priori* reason to expect that the trade-off of consumption across time is the same for all states.

Second, we consider substitutability of consumption across states. One unit of consumption at a time in a particular state of nature may be a close substitute to one unit of consumption at the same time in another state of nature for an individual. But there is no economic reason to expect that *all* individuals with continuous preferences consider consumptions in different states to be close substitutes. Thus the topology on $\mathbf{X}$ should not *a priori* build in substitutability of consumption across states in an arbitrary manner. On the other hand, it is quite intuitive that one unit of consumption in a collection of states or in an *event* which is very unlikely to occur should be close to not consuming at all.

With these considerations in mind, a natural way of aggregating the path-wise construction in the previous paragraph is to just take expectation. Taking expectations embodies the notion that consumptions at two disjoint events are perfect nonsubstitutes except when both events are negligible in probability and thus the two consumption patterns are almost indistinguishable from zero. Taking expectations also embodies the notion that, at any time, the differences of preferences for consumption in two events with the same probability is a decreasing function in the degree of overlap of the two events. We here remind the reader that defining such a strong topology and considering continuous preferences certainly does not rule out preferences that actually consider consumptions across states to be close substitutes. They will be special cases of preferences continuous in the "strong" topology. In the same L^p example above, except now that p is nonrandom, the norm of a consumption pattern $x \in \mathbf{X}_+$ is then $\left(\mathbf{E}\left[\int_0^1 |x(t)|^p dt + |x(1)|^p\right]\right)^{\frac{1}{p}}$, which is a standard L^p norm[4] but on cumulative consumptions rather than on consumption rates.

In section 3, we show that, in fact, preferences continuous in our topologies take an economically appealing view of intertemporal consumption. In particular, consumptions at nearby dates, which can be random, where there are no discontinuities of information are almost perfect substitutes. Furthermore, for reasons to be made precise, consumptions at nearby deterministic dates are always close substitutes even when there might be information surprises. Section 4 characterizes the topological duals of the family of topologies under consideration. We show there that the topological dual spaces corresponding to our topology – spaces which contain the possible shadow prices of consumption – consist of processes that can be decomposed into the sum of two components: an absolutely continuous part and a martingale part. The existence of the martingale part is due to the temporal resolution of uncertainty. It is known that the sample path properties of a martingale are intimately related to the way information is revealed over time. In particular, a martingale can have a discontinuity only at information surprises. Thus the shadow prices for consumption are continuous except possibly at information surprises.

In section 5, we investigate how standard models fare in our setup. We examine some of the prevalent representations in the literature of "non-time-additive" utility functions such as those in Bergman (1985), Constantinides (1990), Heaton (1990), Ryder and Heal (1973) and Sundaresan (1989). Although such representations have elements that capture the effect of past consumption on current utility, we find that most of them imply preferences that are *not* continuous in the sense that we advocate except for some functional forms studied in Heaton (1990). Similar to the results of HH&K, preferences represented by time-additive or non-time-additive felicity functions that include in their arguments the instantaneous consumption rates are continuous in any of our topologies only if they are linear in the consumption rates.

Section 6 examines the implications of our topologies on the prices of long-lived securities in dynamic asset markets free of arbitrage opportunities. We show that in this case the prices over time of long lived securities inherit the properties of the shadow prices of consumption that we discuss in section 4. In particular, we show that in the case of Brownian motion information structures, and absent any arbitrage opportunities, the prices of long lived securities with absolutely continuous cumulative dividend processes will be Itô processes. This result is purely driven by the continuity of preferences. Section 7 contains concluding remarks. Finally, all proofs are in the appendix.

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[4] The reader may have noticed that this expression is not finite for every $x \in \mathbf{X}_+$. Thus we will have to restrict our attention to a subset of $\mathbf{X}_+$ depending on the topology.

2 Formulation

Consider an agent who lives in a world of uncertainty from time 0 to time 1. There is a single consumption good available for consumption at any time $t \in [0,1]$. Uncertainty is modelled by a complete probability space $(\Omega, \mathcal{F}, P)$. Each $\omega \in \Omega$ is a state of nature which is a complete description of one possible realization of all exogenous sources of uncertainty from time 0 to time 1. The sigma-field $\mathcal{F}$ is the collection of events which are distinguishable at time 1 and P is a probability measure on $(\Omega, \mathcal{F})$.

We take as given the time evolution of the agent's knowledge about the states of nature. This is modelled by a *filtration* $\mathbf{F} = \{\mathcal{F}_t; t \in [0,1]\}$, an increasing family of sub sigma-fields of $\mathcal{F}$; that is, $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. Interpret $\mathcal{F}_t$ as the information that the agent has at time t. We assume that the filtration $\mathbf{F}$ satisfies the *usual* conditions: that is $\mathbf{F}$ is complete and right-continuous.[5] For brevity, all processes to appear are measurable and adapted to $\mathbf{F}$ unless mentioned otherwise.[6]

For future use, we now define an *information surprise*.[7] To begin, an optional time is a function $\tau : \Omega \to [0,1]$ with $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in [0,1]$. An optional time can always be interpreted as the first time a certain event happens. The condition $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ says that at any time t, it will be known whether a certain event has happened or not. Note that any deterministic time t is an optional time. It is easily seen that if τ is an optional time, then $(\tau + a) \wedge 1$ is an optional time for all $a > 0$, where $\wedge$ denotes the pointwise minimum. On the other hand, $(\tau - a) \vee 0$ may not be an optional time, where $\vee$ denotes the pointwise maximum. An optional time τ is said to be predictable if there exists a sequence of optional times $\{\tau_n\}$ such that $\tau_n \leq \tau$ a.s. and on the set $\{\tau > 0\}$, $\tau_n < \tau_{n+1} < \tau$ and $\tau_n \nearrow \tau$ a.s. The sequence $\{\tau_n\}$ is said to announce τ. Note that for any deterministic time t we can choose $t - \frac{1}{n}$ for $n \geq 1/t$ to be its announcing sequence. Thus any deterministic time t is predictable.

It is clear that an optional time that is not predictable represents an information surprise as it is the occurrence of an event of which one is not forewarned by an announcing sequence. One might also think that a predictable optional time does not represent a surprise since the timing of this event is announced by its announcing sequence. However, knowing the timing does not necessarily mean knowing the exact "content" of the event. For example, it is often known to the public that some changes of macroeconomic policies will be announced at a deterministic time t although the details of the changes will not be known till the announcement. Since the deterministic t is predictable, the time of the announcement of the changes cannot take one by surprise, but the content of the announcement can. In such a case, $\mathcal{F}_{t-} \neq \mathcal{F}_t$ and we say that there are information surprises at time t.

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[5] $\mathbf{F}$ is *complete* if $\mathcal{F}_0$ contains all P-null sets; and it is right-continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} = \bigcap_{t<s} \mathcal{F}_s$.

[6] A measurable stochastic process Y is a mapping $Y : \Omega \times [0,1] \to \Re$ that is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0,1])$, the product sigma-field generated by $\mathcal{F}$ and the Borel sigma-field of $[0,1]$. For each $\omega \in \Omega$, $Y(\omega, \cdot) : [0,1] \to \Re$ is a *sample path* and for each $t \in [0,1]$, $Y(\cdot, t) : \Omega \to \Re$ is a random variable, which we will usually denote by $Y(t)$. The process Y is said to be adapted to $\mathbf{F}$ if for each $t \in [0,1]$, $Y(t)$ is $\mathcal{F}_t$-measurable.

[7] For the following mathematical definitions of optional times, see Dellacherie and Meyer (1978, chapter IV).

More generally, an information surprise is either an optional time that is not predictable or a predictable optional time τ at which $\mathcal{F}_\tau \neq \vee_n \mathcal{F}_{\tau_n}$, where $\{\tau_n\}$ is the announcing sequence for τ and $\vee_n \mathcal{F}_{\tau_n}$ denotes the smallest sigma-field that contains $\mathcal{F}_{\tau_n}$ for all n[8].[9] The information structure $\mathbf{F}$ is said to be *quasi left-continuous* at an optional time τ with an announcing sequence $\{\tau_n\}$ if $\mathcal{F}_\tau = \vee_n \mathcal{F}_{\tau_n}$. Thus an information surprise is either a nonpredictable optional time or a predictable optional time at which the information structure is not quasi left-continuous.[10]

We represent the life time consumption pattern of the agent by a process x whose sample paths are positive, increasing, and right-continuous where $x(\omega, t)$ denotes the cumulative consumption from time 0 to time t in state ω. The set of such processes is $\mathbf{X}_+$. The linear span of $\mathbf{X}_+$, the space of processes having right-continuous and bounded variation sample paths, will be denoted by $\mathbf{X}$. We restrict our attention to the consumption set and the commodity space, which are subsets of $\mathbf{X}_+$ and $\mathbf{X}$, respectively, defined as follows.

Definition 1 *Let Φ be the collection of functions $\varphi : \Omega \times \Re_+ \to \Re_+$ measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\Re)$ with, almost surely, $\varphi(\omega, 0) = 0$ and $\lim_{z \to +\infty} \varphi(\omega, z) = +\infty$, which are continuous, strictly increasing, convex, and integrable in ω in that $\int_\Omega \varphi(\omega, z) P(d\omega) < \infty \quad \forall z > 0$. For $\varphi \in \Phi$ let $\mathcal{E}^\varphi$ be the vector subspace of $\mathbf{X}$ defined as:*

$$\mathcal{E}^\varphi = \left\{ x \in \mathbf{X} : E\left[\int_0^1 \varphi\Big(\gamma |x(t)|\Big)\, dt + \varphi\Big(\gamma |x(1)|\Big) \right] < \infty \quad \forall \gamma > 0 \right\}, \tag{1}$$

and define the following norm on $\mathcal{E}^\varphi$:

$$\|x\|_\varphi = \inf\left\{ a > 0 : E\left[\int_0^1 \varphi(\frac{|x(t)|}{a})\, dt + \varphi(\frac{|x(1)|}{a}) \right] \le 1 \right\}. \tag{2}$$

Remark 1 *For the fact that (2) defines a norm on $\mathcal{E}^\varphi$, see Musielak (1983, theorem 1.5). For simplicity, we will use $\mathcal{T}_\varphi$ to denote the topology on $\mathcal{E}^\varphi$ generated by $\|\cdot\|_\varphi$.*

Remark 2 *The condition that $\int_\Omega \varphi(\omega, z)\, P(d\omega) < \infty$ ensures that all the sure consumption patterns are elements of the commodity space $\mathcal{E}^\varphi$.*

The spaces $\mathcal{E}^\varphi$ and associated topologies are natural and simple generalizations of the familiar L^p spaces defined on cumulative consumption. In contrast, existing dynamic models of asset prices such as Duffie and Zame (1989) and Huang (1987) use L^p topologies defined on consumption "rates", which are essentially equivalent to the total variation topology on cumulative consumption and are much stronger topologies.

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[8] Formally, $\mathcal{F}_\tau$ is the sigma-field of events prior to τ consisting of all events $A \in \mathcal{F}$ such that $A \cap \{\tau \le t\} \in \mathcal{F}_t$ for all $t \in [0, 1]$.

[9] For more discussions on the classification of optional time and its relation to "surprises" see Huang (1985).

[10] The filtration generated by a diffusion process, a Poisson process, or a combination of the two is quasi left-continuous at all predictable optional times once the filtration is completed. Moreover, it has been shown by Meyer (1963) that any information structure generated by a process that is continuous at predictable optional times (defined with respect to its natural filtration) and has the strong Markov property is quasi left-continuous.

Henceforth we will take $\mathcal{E}^\varphi$ for some $\varphi \in \Phi$ to be the commodity space of the economy that we study. The agent in the economy has a preference relation $\succeq$ defined on $\mathcal{E}^\varphi_+$, where $\mathcal{E}^\varphi_+$ denotes the positive orthant of $\mathcal{E}^\varphi$ and is the agent's consumption set. We assume that $\succeq$ is continuous with respect to $\mathcal{T}_\varphi$. We will demonstrate in the analysis to follow that $\succeq$ exhibits economically appealing properties.

3 Properties of Continuous Preferences

In this section we show that the topology $\mathcal{T}_\varphi$ gives economically reasonable sense of "closeness" for consumption patterns in $\mathcal{E}^\varphi_+$. In particular, $\succeq$ treats a fixed quantity of consumption at nearby predictable optional times as almost perfect substitutes.[11] In addition, $\succeq$ regards consumption patterns that differ slightly in the quantity and the timing of consumption as close commodities. We will accomplish this objective by establishing the convergence of several sequences in three propositions. These convergent sequences should be understood in economic terms as follows: Let $x_n, x \in \mathcal{E}^\varphi_+$ and $x_n \to x$ in $\mathcal{T}_\varphi$ as $n \to \infty$. Then by the continuity of $\succeq$, $x_n \succeq y$ for all n implies $x \succeq y$ and $y \succeq x_n$ for all n implies $y \succeq x$.

First, we show, roughly, that two patterns of consumption which have almost equal cumulative consumption at every point in time and in each state are close.

Proposition 1 *Let $x, y, x_n \in \mathcal{E}^\varphi_+$ for all n. If $\lim_{n\to\infty} \sup_{t\in[0,1]} |x_n(\omega,t) - x(\omega,t)| = 0$ a.s. and if there exists a random variable K with the property that $\mathrm{E}[\varphi(K)] < \infty$, such that $\sup_{t\in[0,1]} |x_n(\omega,t) - x(\omega,t)| \le K(\omega)$ a.s., then $x_n \to x$ in $\mathcal{T}_\varphi$ as $n \to \infty$.*

Second, we show that a feasible delay or an advancement of a sizable quantity of consumption for a small enough time is regarded as insignificant. The information constraint is crucial here. Note that it is always feasible to delay consuming. A feasible advancement of consumption, however, is one which could be arranged within the constraint that all consumption patterns be adapted to **F**. One unit of consumption at time t in an event $A \in \mathcal{F}_t$ may not be advanced to time $t - \epsilon$ in the same event for $\epsilon > 0$ however small, since A may not be a distinguishable event in $\mathcal{F}_{t-\epsilon}$ and hence the proposed change in the consumption pattern may not be feasible. On the other hand, one unit of consumption at time t in an event can always be delayed to any time $s > t$ in the same event. The following proposition formalizes this discussion.

Proposition 2 *Let $x_n = k\chi_{[(\tau+\frac{1}{n})\wedge 1,1]}$ and $x = k\chi_{[\tau,1]}$, where $k > 0$ is a scalar, τ is an optional time, and χ is an indicator function. Then x and $x_n \in \mathcal{E}^\varphi_+$, for all n, and $\|k\chi_{[(\tau+\frac{1}{n})\wedge 1,1]} - k\chi_{[\tau,1]}\|_\varphi \to 0$ as $n \to \infty$. On the other hand, suppose that $\{\tau_n\}$ is a sequence of optional times with $\tau_n \le \tau$, and on the set $\{\tau > 0\}$, $\tau_n < \tau_{n+1} < \tau$. Putting $x'_n = k\chi_{[\tau_n,1]}$, $\|x'_n - x\|_\varphi \to 0$ as $n \to \infty$ if and only if $\{\tau_n\}$ is an announcing sequence for τ, and hence τ is predictable.*

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[11] The astute reader would realize that this implies that a fixed quantity of consumption at nearby deterministic times are close substitutes even if there are information surprises. This, as the reader will see, is an intuitive result.

Note that, in line with our discussion before this proposition, $\chi_{[\tau,1]}$ can always be approximated closely by $\chi_{[(\tau-\frac{1}{n})\vee 0,1]}$ in the norm for large n, but unfortunately $(\tau-\frac{1}{n})\vee 0$ may not be an optional time and thus $\chi_{[(\tau-\frac{1}{n})\vee 0,1]}$ may not be a feasible consumption pattern.

Note also that since any deterministic t is predictable, the second assertion of this proposition shows that one unit of sure consumption at time t is a close substitute of one unit of sure consumption at $t-\frac{1}{n}$ when n is large. This together with the first assertion of this proposition implies that sure consumptions at nearby dates are close substitutes. It is worthwhile to comment on the interpretation of this result in situations when the information structure is not quasi left-continuous at t.

Proposition 2 implies that one unit of consumption for sure at t is a close substitute to one unit of consumption for sure at $t-\frac{1}{n}$ for large n even if there is some announcement of policy changes at time t and the details of this announcement are not known till t. One can explain this intuitively by observing that instants before t, the "average" or "expected" content of the announcement will be known. For example, market participants might anticipate tightening of money supply, but the exact amount of the new money supply is not known.

The actual announcement at time t could be either high reduction in money supply, which is perceived to be relatively "bad" news, or a low reduction in money supply, which is relatively "good" news given the anticipations instants before t. One unit of consumption at time t in the event of good news may be preferred to one unit of consumption instants before, which in turn may be preferred to one unit of consumption at t in the event of bad news. Considering the possibility of both "bad" and "good" news, agents find on the balance that one unit of consumption at t is a close substitute for one unit of consumption at time $t-\frac{1}{n}$, for large n, even when there are unanticipated events at t. The same intuition carries over to a general predictable optional time τ with an announcing sequence $\{\tau_n\}$. Here one unit of consumption at τ is a close substitute to one unit of consumption at τ_n for large n even if $\mathcal{F}_\tau \neq \bigvee_n \mathcal{F}_{\tau_n}$.

We conclude our investigation of the flexibility provided by the topology $\mathcal{T}_\varphi$ in allowing substitution of consumption between adjacent times by analyzing a rather more general trade-off between the quantity and the times of consumption. In proposition 2, we showed that delaying or advancing a *fixed* amount of consumption for a small enough period results in a very small change in total satisfaction. In the following analysis, consumption patterns differ slightly not only in the timing of consumption but also in the amounts consumed.

Let $x, y \in \mathcal{E}_+^\varphi$ and define a Prohorov metric[12] path-by-path:

$$\rho_\omega(x,y) = \inf\{\epsilon \geq 0 : x(t+\epsilon,\omega)+\epsilon \geq y(t,\omega) \geq x(t-\epsilon,\omega)-\epsilon, \forall t \in [0,1]\}.$$

If $y(\omega,\cdot)$ is within ϵ of $x(\omega,\cdot)$ in the Prohorov sense, then at every time t in state ω, the total consumption under x is within ϵ of the total consumption under y at some time no more than ϵ away from t. At nonpredictable times, however, Proposition 2 implies that as $\epsilon \to 0$, the ϵ neighborhood of x does not contain any y that advances consumption to earlier time since that violates the information constraint.

The idea is that if $\rho_\omega(x,y)$ is uniformly smaller than ϵ across states of nature, then sizable differences of consumption between x and y at predictable optional times can only occur in a time interval less than ϵ in width. Such differences should not drastically change an agent's utility if x were exchanged for y. As it turns out, there is a stronger result available and $\rho_\omega(x,y)$ need not be uniformly small. It is sufficient that $\rho_\omega(x,y)$ be integrably small and hence $\rho_\omega(x,y)$ can be large on a set of small probability.

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[12] See, for example, Billingsley (1968, pp. 237-238).

Proposition 3 *Let $x, x_n \in \mathcal{E}_+^\varphi$ with the property that there exists an $\mathcal{F}$-measurable function, say K, such that $\rho_\omega(x_n, x) \leq \frac{K(\omega)}{n}$ P-a.s. and $\mathrm{E}[\varphi(K)] < \infty$. Then $\|x_n - x\|_\varphi \to 0$ as $n \to \infty$. Thus $x_n \succeq y$ for all n implies that $x \succeq y$ and $y \succeq x_n$ for all n implies that $y \succeq x$.*

4 Duality

In standard general equilibrium theory of the Arrow-Debreu sort and in the theory of arbitrage pricing of financial assets, prices for consumption at different dates in different states, or the *shadow prices for consumption*, come from the space of linear functionals continuous in the topology with respect to which agents' preferences are assumed to be continuous; for the former see, e.g., Mas-Colell (1986), and for the latter see Kreps (1981) and Ross (1978). This space of continuous linear functionals is called the *topological dual space*.[13] In particular, Mas-Colell (1986) showed that there always exists an equilibrium in a representative agent pure exchange economy with prices lying in the topological dual if the agent's preferences satisfy a strong monotonicity condition. For our analysis here, we will assume that shadow prices for consumption indeed come from the topological dual space and characterize this space.

Let ψ be a continuous linear functional that gives equilibrium prices. Then $\psi(x)$ is the price at time 0 of the consumption claim x. Given that ψ is a continuous linear functional, the time 0 prices for the consumption patterns identified to be close substitutes in Propositions 1 to 3 will certainly be close. Much more can be said, however.

We will find out soon that ψ can be represented in the form $\psi(x) = \mathbf{E}\left[\int_{0-}^{1} f(t)dx(t)\right]$, where, roughly, f is a process that is decomposable into two parts: a process having absolutely continuous sample path and a martingale.[14] We interpret $f(\omega, t)$ to be the time 0 price of one unit of consumption at time t in state ω, per unit of probability, or simply the shadow price for time t consumption in state ω. Since the sample path properties of a martingale are solely determined by how information about the state of nature is revealed over time, we can then characterize sample path properties of f from studying the properties of the information structure $\mathbf{F}$.

We now proceed to characterize the topological dual of $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$, denoted by $\mathcal{E}^{\varphi^*}$. Some definitions are in order.

Definition 2 *A function $\varphi \in \Phi$ is said to be an N-function if*

$$\lim_{z \downarrow 0} \frac{\varphi(\omega, z)}{z} = 0 \quad \textit{and} \quad \lim_{z \uparrow \infty} \frac{\varphi(\omega, z)}{z} = \infty \quad \textit{for all} \quad \omega \in \Omega\,.$$

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[13] More formally, $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$ is a topological linear space and the space of continuous linear functionals on $(\mathcal{E}^\varphi, \mathcal{T}_\varphi)$ is its topological dual.

[14] A martingale Y is a process (adapted to $\mathbf{F}$) that has right-continuous sample paths with the property that $\mathrm{E}[Y(s)|\mathcal{F}_t] = Y(t)$ *a.s.* for all $s \geq t$.

By convexity, for φ *an N-function, we can write* $\varphi(\omega,|z|) = \int_0^{|z|} v(\omega,s)\,ds$, *where* $v(\omega,s)$ *is the right-hand derivative of* $\varphi(\omega,s)$ *for a fixed* ω. *Let* $v^*(\omega,\sigma) = \sup\{s: v(\omega,s) \le \sigma\}$, *then* $\varphi^*(\omega,z) \equiv \int_0^{|z|} v^*(\omega,\sigma)\,d\sigma$ *is said to be complementary to* φ.

Example 1 *(1) If* $\varphi(\omega,z) = |z|^{p(\omega)}, 1 < p(\omega) < \infty$, *then* $\varphi^*(\omega,y) = |y|^{q(\omega)}$, *where* $\frac{1}{p(\omega)} + \frac{1}{q(\omega)} = 1$. *(2) If* $\varphi(\omega,z) = e^z - z - 1$, *then* $\varphi(\omega,y) = (1+y)ln(1+y) - y$.

The following proposition describes $\mathcal{T}_\varphi$-continuous linear functionals on the space $\mathcal{E}^\varphi$. We first record two definitions.

Let $\mathcal{O}$ be the sigma-field on $\Omega \times \mathcal{F}$ generated by all the right-continous and adapted process. It is known that any process, as a mapping from $\Omega \times \mathcal{F}$ to $\Re$, measurable with respect to $\mathcal{O}$ is adapted; see, Chung and Williams (1990, p.27). We use $\mathbf{O}$ to denoe the space of optional processes.

Proposition 4 *Let* $\varphi \in \Phi$ *be an N-function with a complementary function* φ^*. *Then* $\psi : \mathcal{E}^\varphi \to \Re$ *is a* $\mathcal{T}_\varphi$*-continuous linear functional if and only if there exists an adapted process* f *with absolutely continuous sample paths, and with*

$$f' \in L^{\varphi^*} \equiv \left\{ x \in \mathbf{O}: E\left[\int_0^1 \varphi^*(\gamma|x(t)|)\,dt + \varphi^*(\gamma|x(1)|)\right] \to 0 \quad as \quad \gamma \downarrow 0 \right\},$$

so that

$$\psi(x) = \mathbf{E}\left[\int_{0-}^1 g(t)dx(t)\right] \quad \forall x \in \mathcal{E}^\varphi,$$

where $g(t) = f(t) + m(t)$, *and* $m(t) = \mathbf{E}[-f'(1) - f(1)|\mathcal{F}_t]$ *for all* $t \in [0,1]$, *is a martingale, or equivalently,* $g(t) = f(t) - f'(1) - f(1)$, *which is a process with absolutely continuous paths but may not be adapted.*

An important special case of Proposition 4 is when $\varphi(\omega,z) = |z|^p$ with $1 < p < \infty$. The case when $p = 1$ is not covered in Proposition 4, however, since $|z|$ is not an N function. The following proposition, whose proof is left for the reader, gives the duality result for this L^p family.

Proposition 5 *Let* $\varphi(\omega,z) = |z|^p$, *for all* ω, *with* $1 \le p < \infty$. *Let* q *be such that* $1/p + 1/q = 1$. *Then* $\psi : \mathcal{E}^\varphi \to \Re$ *is a* $\mathcal{T}_\varphi$*-continuous linear functional if and only if there exists an adapted process* f *with absolutely continuous sample paths, with*

$$(f', f'(1)) \in L^q(\Omega \times [0,1], \mathcal{O}, P \times \lambda) \times L^q(\Omega, \mathcal{F}, P),$$

so that

$$\psi(x) = \mathbf{E}\left[\int_{0-}^1 g(t)dx(t)\right] \quad \forall x \in \mathcal{E}^\varphi,$$

where $g(t) = f(t) + m(t)$ *and* $m(t) = \mathbf{E}[-f'(1) - f(1)|\mathcal{F}_t]$ *for all* $t \in [0,1]$, *is a martingale, or equivalently,* $g(t) = f(t) - f'(1) - f(1)$, *which is a process with absolutely continuous paths but is not necessarily adapted.*

Note that from the above proposition, when $p = 1$, the dual space is composed of processes that are sums of processes with Lipschitz continuous paths and bounded martingales. Another case of the function φ which is not covered in the above propositions is when φ is asymptotically linear for some ω's, in that for some $\omega \in \Omega$, we have $\lim_{z\to\infty} \frac{\varphi(\omega,z)}{z} = \alpha > 0$. In the case of certainty, HH&K have shown that the topological dual in this case is the space of Lipschitz continuous functions. We obtain a similar characterization for the case of uncertainty under the assumption that the function φ is "integrably asymptotically linear." This result is recorded in the following proposition.

Proposition 6 *Let the function φ be integrably asymptotically linear, in that for any $\epsilon > 0$, there exists a scalar $\alpha > 0$ and a random variable K, with the property that $\mathrm{E}[\varphi(K)] < \infty$, such that $\varphi(\omega, z) \le \alpha z + \epsilon$ for all $z \ge K(\omega)$ a.s. Then $\mathcal{E}^{\varphi}$ and $\mathcal{T}_{\varphi}$ are the same as the consumption space and the topology constructed using $\varphi(\omega, z) = |z|$ for all $\omega \in \Omega$. Moreover, an element of the topological dual space of $\mathcal{E}^{\varphi}$ is the sum of a process with Lipschitz continuous sample paths and a bounded martingale.*

In the case of certainty, a continuous linear functional can be represented essentially by an absolutely continuous function (see HH&K or note that under certainty, a martingale must be a constant). Here, however, a continuous linear functional is represented by a process that is the sum of an absolutely continuous process and a martingale. It is known that the sample path properties of a martingale are determined by the way information is revealed over time: A martingale can have a discontinuity only at nonpredictable optional times or at predictable optional times where the information structure is not quasi left-continuous; see, for example, Meyer (1966, Theorem VI.14). It then follows that shadow prices for consumption at nearby random adjacent dates are almost equal when there are no "surprises."

An information structure $\mathbf{F}$ is said to be continuous if $P(A|\mathcal{F}_t) \equiv \mathrm{E}[\chi_A|\mathcal{F}_t]$ *a.s.* is a process with almost all its sample paths continuous for all $A \in \mathcal{F}$; that is, the posterior probability of any event evolves continuously over time. It is clear that a continuous information structure must be quasi left-continuous at all optional times. It is also known that an information structure is continuous if and only if all optional times are predictable; see Huang (1985a, theorem 6.3). Thus when $\mathbf{F}$ is continuous, $\mathcal{E}^{\varphi^*}$ consists only of continuous processes and thus shadow prices of consumption at nearby random dates are almost equal. It is also known that if a martingale is continuous on a stochastic interval, it is either a constant or of unbounded variation there (Fisk (1965)). Hence there are cases where the shadow prices for consumption at nearly adjacent dates are almost equal but fluctuate in a nowhere differentiable fashion. This phenomenon is due to temporal resolution of uncertainty. When there is no uncertainty, the martingale part of the prices for consumption becomes a constant and prices degenerate to absolutely continuous functions of time.

The characterization of shadow prices in proposition 4 can help us better understand the results of proposition 2 in which one unit of consumption at any time t is a close substitute for one unit of consumption at times $t - \frac{1}{n}$, for large n, even when there are unanticipated events at t. Suppose that the prices are represented by $g \in \mathcal{E}^{\varphi^*}$ with $g(t) = f(t) + m(t)$, where f is a process with absolutely continuous sample paths and m is a martingale. Let τ be a predictable optional time with an announcing sequence $\{\tau_n\}$ such that $\mathcal{F}_\tau \neq \bigvee_n \mathcal{F}_{\tau_n}$ and suppose that

$m(\tau) \neq \lim_n m(\tau_n)$ with a strictly positive probability. Since one unit of consumption at τ is a close substitute to one unit of consumption at τ_n for large n, we expect their time zero price to be close despite the discontinuity of m at τ. The time zero values of one unit of consumption at τ_n and at time τ are, respectively, $\mathbf{E}[f(\tau_n) + m(\tau_n)] = \mathbf{E}[f(\tau_n)] + m(0)$ and $\mathbf{E}[f(\tau)+m(\tau)] = \mathbf{E}[f(\tau)]+m(0)$. Closeness of the prices is a consequence of the fact that m is a martingale and f has continuous paths. The effects of the discontinuities of m that correspond to "surprises" cancel each other out in expectations.

An important special case of continuous information structure is when $\mathbf{F}$ is generated by a Brownian motion. In this case, the dual space contains Itô processes only. This characterization is given in the following proposition:

Proposition 7 *Let* $\mathbf{F}$ *be generated by a Brownian motion and let* g *be an element of any of the dual spaces characterized in Propositions 4, 5 and 6, then* g *is an Itô process:*

$$g(t) = \int_0^t f'(s)\, ds + \int_0^t \theta(s)^{\top}\, dw(s),$$

where f *is a process with absolutely continuous paths and* f' *denotes its derivative with respect to time, where* w *is a Brownian motion and where* θ *is adapted to* $\mathbf{F}$ *and* $\int_0^1 |\theta(t)|^2\, dt < \infty$ *a.s.*

Note that in the existing dynamic models of asset prices in pure exchange economies in continuous time such as Duffie and Zame (1989) and Huang (1987), agents learn the true state of nature by observing a Brownian motion and shadow prices for consumption over time are represented by an Itô process. Agents' preferences there are defined only over consumption rates and are continuous with respect to the L^p topology on the rates. Since these L^p topologies on consumption rates are much stronger topologies than $\mathcal{T}_\varphi$, their topological duals contain processes that are not Itô processes. Thus Duffie and Zame and Huang assumed that agents' preferences were time-additive and state-independent and that the aggregate endowment process was an Itô process. Here, however, we have a much weaker topology that treats consumption at nearby dates as close substitutes except possibly at information surprises. Hence shadow prices for consumption are Itô processes independent of the special properties of the aggregate endowment process; see, Proposition 7.

Finally, an example of a discontinuous information structure is when information is obtained by observing the realizations of a simple Poisson $\{N(t); t \in [0,1]\}$ with intensity λ. In this case, a martingale can be represented by a stochastic integral with respect to the Poisson martingale, $N(t) - \lambda t$, and thus may have a discontinuity only when the Poisson process jumps; see, e.g., Davis (1973). As a consequence, the shadow prices for consumption can have a discontinuity only at a "surprise" which occurs when the Poisson process jumps.

5 On Standard Models

In this section we examine how the standard models fare in the context of our current model. Most of the specifications of preferences in the literature are given by the following construction. let x be a consumption plan. Consider an adapted process $\theta(\omega, t)$, which is uniformly bounded across ω and t, which is differentiable in t, and whose derivatives are uniformly bounded over ω. An example of such a process is the Gauss kernel given by $\theta(\omega, t) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-t^2}{2\sigma^2}}$.

Now using the process θ, construct from x another process $\hat{x}$ which gives a path-by-path weighted average of recent past consumption. In other words, put $\hat{x}(\omega, t) = \int_0^t \theta(\omega, s)\, dx(\omega, t - s)$. Let v be a felicity function which takes as arguments the current consumption "rate" $x'(t)$ together with the "smoothed" earlier consumption $\hat{x}(t)$. Thus one represents preferences by:

$$V(x) = \mathbf{E}[\int_0^1 v(x', \hat{x}, t)\, dt]. \tag{3}$$

Examples of such representations include Bergman (1985), Constantinides (1990), Heaton (1990), Ryder and Heal (1973), and Sundaresan (1989). The standard time-additive representation is, of course, a special case of (3). Inclusion of the "smoothed" recent consumption in the instantaneous felicity function v captures the effect of past consumption on one's current satisfaction. However, including current consumption in the felicity function destroys the continuity of preferences in the sense we argue for in this paper except for certain special cases. We record this fact in the following proposition.

Proposition 8 *Let* $v(c, z, t)\colon \Re_+^2 \times [0, 1]$ *be jointly continuous. The utility function* $V\colon \mathcal{E}_+^{\varphi} \to \Re$ *defined in (3) is continuous in* $\mathcal{T}_\varphi$ *only if there exists jointly continuous functions* $\alpha\colon \Re_+ \times [0, 1] \to \Re$ *and* $\beta\colon \Re_+ \times [0, 1] \to \Re$ *such that* $v(c, z, t) = \alpha(z, t)c + \beta(z, t)$.

An immediate consequence of Proposition 8 is that any felicity function $v(c, z, t)$ that is strictly concave in c cannot represent preferences continuous in $\mathcal{T}_\varphi$. For example, in the habit formation models of Constantinides (1990) and Sundaresan (1989), the felicity functions are strictly concave functions of $c - z$ and thus the preferences they represent do not treat consumption at nearby dates as close substitutes.

We propose an alternative representation of the utility, which keeps the spirit of the standard model and gives rise to preferences continuous in our topologies. To achieve this objective, we express the utility as an integral of felicity as in the standard model. However, the felicity at time t depends only on a weighted average of the consumption in the "recent past." Let $u(\hat{x}, t)$ be a felicity function which is state-independent, continuous and concave in $\hat{x}$. Let

$$U(x) = \mathbf{E}[\int_0^1 u(\hat{x}, t)\, dt]. \tag{4}$$

The fact that under these conditions, the utility function given in (4) is continuous in any of our topologies is recorded in the following proposition.

Proposition 9 *Let the averaging process* θ *and the felicity function* u *satisfy the conditions described above. Let* $U(x)$ *be constructed as in (4). Preferences represented by* U *are continuous in any* $\mathcal{T}_\varphi$.

Remark 3 *Note that we allow* $U(x) = -\infty$ *for some* $x \in \mathcal{E}_+^{\varphi}$. *This can occur if* u *is unbounded from below.*

6 Arbitrage Pricing of Financial Assets

Although we have not established existence of an Arrow-Debreu equilibrium in our model, we can still provide useful results about the behavior of financial assets using the notion of the absence of arbitrage possibilities. Using a similar commodity space and a Mackey topology generated by the space of bounded and continuous processes and barring arbitrage opportunities, Huang (1985b) has shown that security prices *over time* are continuous except possibly at surprises or at lump-sum ex-dividend dates. In particular, price processes are generalized Itô processes between lump-sum ex-dividend dates if the information structure is generated by a Brownian motion. However, requiring the dual of the commodity space to have only continuous processes is a bit too strong. It implies that consumption at nearby dates are almost perfect substitutes even at times of surprise. Using the family of economically more reasonable topologies we advanced, we get essentially all the results of Huang (1985b) in the following manner.

Assume that the commodity space $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is separable.[15] There are N securities which are available for trading any time in $[0,1]$. Security n is denoted by its cumulative dividend process $x_n \in \mathcal{E}^{\varphi}$. Let $S_n(t)$ denote the ex-dividend price process for security n at time t. Since securities are traded ex-dividends, we assume that $x_n(0) = 0$ for all n and $S_n(1) = 0$; that is, there is no cumulative dividend at time 0 and the ex-dividend price of a security at time 1 is zero. A trading strategy θ is an N-dimensional process that prescribes the portfolio strategy for the traded securities. Without getting into technical details, we shall only allow agents to use *simple trading strategies* – strategies that are adapted, bounded, left-continuous, and change their values at a finite number of non-random time points.[16] We shall say that $m \in \mathcal{E}^{\varphi}$ is *marketed* if there is a strategy θ so that

$$m(t) - m(0) = \theta(0)^{\top} S(0) + \int_0^{t-} \theta(s)^{\top} dS(s) + \int_0^{t-} \theta(s)^{\top} dx(s),$$

where $^{\top}$ denotes "transpose," $x(t) = (x_1(t), \ldots, x_N(t))^{\top}$, and the integrals are defined path-by-path; alternatively, we say that m is *financed* by θ. Let M denote the space of marketed consumption patterns. It is clear that M is a linear subspace of $\mathcal{E}^{\varphi}$. An element $m \in M$ financed by θ is a *simple free lunch* if $\theta(0)^{\top} S(0) \leq 0$, $m \in \mathcal{E}^{\varphi}_{+}$, and $m \neq 0$. Barring simple free lunches, each $m \in M$ has a unique price at time 0, $m(0) + \theta(0)^{\top} S(0)$, where θ finances m, and we can define a linear functional on M by $\pi(m) = m(0) + \theta(0)^{\top} m(0)$.

Now we show that the linear functional π has a nice representation if the securities market does not admit *free lunches*, a concept due to Kreps (1981). A *free lunch* is a sequence $\{(m_n, x_n) \in M \times \mathcal{E}^{\varphi}; n = 1, 2, \ldots\}$ and a bundle $k \in \mathcal{E}^{\varphi}_{+}$ with $k \neq 0$ such that $m_n - x_n \in \mathcal{E}^{\varphi}_{+}\ \forall n$, $\|x_n - k\|_{\varphi} \to 0$, and $\liminf_n \pi(m_n) \leq 0$. Suppose that there are no free lunches. Theorem A.1 of Duffie and Huang (1986) shows that there exists an extension of π to all of $\mathcal{E}^{\varphi}$, ψ, strictly positive in that $\psi(x) > 0$ if $x \in \mathcal{E}^{\varphi}_{+}$ and $x \neq 0$.

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[15] $(\mathcal{E}^{\varphi}, \mathcal{T}_{\varphi})$ is a separable topological vector space if $(\Omega, \mathcal{F}, P)$ is separable.

[16] To consider strategies more general than simple strategies, we first need to discuss how general stochastic integrals are defined, which we do not want to do here. Interested readers should consult Huang (1985b) for a discussion in a context similar to our setup here.

Let φ satisfy conditions of Propositions 4, 5, or 6, then $\psi(x) = \mathbf{E}\left[\int_{0-}^{1} g(t)dx(t)\right]$ for all $x \in \mathcal{E}^{\varphi}$, for some strictly positive process g, which is the sum of a martingale and a process of absolutely continuous sample paths. Let $S_m(t)$ be the ex-dividend price at time t of $m \in M$. It follows from Proposition 5.1 of Huang (1985b) that

$$S_m(t) = \frac{\mathbf{E}\left[\int_0^1 g(s)dm(s)|\mathcal{F}_t\right] - \int_0^t g(s)dm(s)}{g(t)}. \tag{5}$$

Note that the first term of the numerator of (5) is a martingale and thus the properties of g translate into those of S_m in a natural way. For example, between discontinuities of m, S_m can have discontinuities only at surprises. In particular, suppose that **F** is generated by a Brownian motion and m is an absolutely continuous process. Since a martingale then can be represented by an Itô integral and the second term in the numerator of (5) is an absolutely continuous process, the numerator of (5) is an Itô process. The denominator is a martingale plus a process of absolutely continuous paths and thus is an Itô process. Since g is strictly positive, Itô's lemma implies that S_m is an Itô process. These are all economically appealing properties of price processes of securities determined by preferences and information flow and independent of endowments.

7 Concluding Remarks

In this paper we advanced a family of topologies defining closeness between consumption patterns over time under uncertainty to capture the intuitive idea that consumptions at nearby dates are almost perfect substitutes. The idea we tried to formalize is certainly not new. Our contribution is defining the topologies and, more important, characterizing the topological dual spaces. We feel that our choice of topologies is appropriate for the following two reasons. First, in the degenerate case where the true state of nature is revealed at time 0, our conclusion agrees with the results under certainty of HH&K. Second, in the nondegenerate case, the topological duals are natural generalizations of those under certainty. In this case, shadow prices of consumption are processes that can be decomposed as the sum of two components: a process of absolutely continuous sample paths and a martingale. In the case of uncertainty a new element, the pattern of information flow, affects the sample path properties of the shadow price process of consumption. This effect is captured in the martingale component of the process. It is known that a martingale can make discontinuous changes only at surprises. Thus the shadow prices for consumption are continuous except possibly at surprises. This is an intuitively appealing result which holds regardless of the nature of the endowment process.

The proposed family of topologies, however, does not give rise to certain mathematical properties know[illegible] be sufficient for the existence of an Arrow-Debreu equilibrium.[17] We hope tha[illegible]rther research produces more powerful existence theorems which can accommodate our mo[illegible] In any case, provided that there exists an Arrow-Debreu equilibrium, one can implement the Arrow-Debreu equilibrium by continuously trading a few long-lived securities as in Duffie (1986) and Duffie and Huang (1985), and thus establish the existence of a dynamic equilibrium with dynamically complete markets.

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[17] For example, the commodity space is not a topological vector lattice, which we would want if we used Mas-Colell (1986), and the topological dual is not a lattice in the order dual, which we would want if we used Mas-Colell and Richard (1990). For proofs of the above two assertions see Hindy and Huang (1989).

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Appendix A Proofs

In all the proofs, λ^* denotes the measure on $[0,1]$ equal to the Lebesgue measure plus a point mass at 1.

PROOF OF PROPOSITION 1: It suffices to show that for any $\gamma > 0$

$$\mathbf{E}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\, \lambda^*(dt)\right] \to 0.$$

The assertion then follows from the Lebesgue dominated convergence theorem and the fact that $\varphi(\omega, z)$ is continuous in z and $\varphi(\omega, 0) = 0$. ∎

PROOF OF PROPOSITION 2: First, we show that $k\chi_{[\tau,1]} \in \mathcal{E}_+^{\varphi}$ for all optional time τ and all $k > 0$. Note that

$$\mathbf{E}\left[\int_0^1 \varphi(\gamma k \chi_{[\tau,1]}(t))\lambda^*(dt)\right] \le 2\mathbf{E}\left[\varphi(\gamma k)\right] < \infty,$$

where the first inequality follows from the fact that φ is strictly increasing and the second inequality follows from the hypothesis that φ is integrable.

Second, the fact that $(\tau + \frac{1}{n}) \wedge 1$ is an optional time whenever τ is can be seen by directly checking the definition.

Third, the hypothesis that φ is integrable and Lebesgue convergence theorem imply that

$$\begin{aligned} & \lim_{n\to\infty} \mathbf{E}\left[\int_0^1 \varphi(\gamma k(\chi_{[\tau,1]}(t) - \chi_{[(\tau+\frac{1}{n})\wedge 1,1]}(t)))\lambda^*(dt)\right] \\ = {} & \mathbf{E}\left[\int_0^1 \lim_{n\to\infty} \varphi(\gamma k(\chi_{[\tau,1]}(t) - \chi_{[(\tau+\frac{1}{n})\wedge 1,1]}(t)))\lambda^*(dt)\right] \\ = {} & \mathbf{E}\left[\int_0^1 \varphi(\gamma k \lim_{n\to\infty}(\chi_{[\tau,1]}(t) - \chi_{[(\tau+\frac{1}{n})\wedge 1,1]}(t)))\lambda^*(dt)\right] = 0, \end{aligned}$$

where the first and second equalities follow from the fact that $\varphi(\omega, z)$ is continuous in z and is zero at $z = 0$, respectively. Thus, by the continuity of $\succeq$, $x_n \succeq y$ for all n implies $x \succeq y$ and $y \succeq x_n$ for all n implies that $y \succeq x$.

Fourth, suppose that $\{\tau_n\}$ is a sequence of optional times with $\tau_n \le \tau$, and $\tau_n < \tau$ and $\tau_n < \tau_{n+1}$ on the set $\{\tau > 0\}$ and that $k\chi_{[\tau_n,1]} \to k\chi_{[\tau,1]}$ in $\|\cdot\|_\varphi$. This implies that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^1 \varphi(\gamma k(\chi_{[\tau,1]}(t) - \chi_{[\tau_n,1]}(t)))\lambda^*(dt)\right] \to \infty$$

for all $\gamma > 0$. This implies that $k\chi_{[\tau_n,1]} \to k\chi_{[\tau,1]}$ in $P \times \lambda^*$-measure, and hence there exists a subsequence $\{\tau_{n_i}\}$ such that

$$\lim_{n_i\to\infty} \chi_{[\tau_{n_i},1]}(\omega, t) - \chi_{[\tau,1]}(\omega, t) = 0 \quad P \times \lambda^* - a.e.,$$

which implies that $\tau_{n_i} \to \tau$ *a.s.*. Since $\{\tau_n\}$ is an increasing sequence, this implies that that $\tau_n \to \tau$ *a.s.* and thus τ is predictable with $\{\tau_n\}$ an announcing sequence.

The proof for the last assertion is just the reverse of the above paragraph. ∎

PROOF OF PROPOSITION 3: We omit the proof since it is similar to the proof of Proposition 5 of Hindy, Huang, and Kreps (1991). ∎

PROOF FOR PROPOSITION 4: Integration by parts path-by-path gives

$$\begin{aligned}\psi(x) &= \mathbf{E}\left[\int_{0-}^{1} g(t)dx(t)\right]\\ &= \mathbf{E}\left[\int_{0-}^{1} f(t)dx(t) + \int_{0-}^{1} m(t)dx(t)\right]\\ &= \mathbf{E}\left[f(1)x(1) - \int_{0}^{1} x(t)f'(t)dt + m(1)x(1)\right]\\ &= \mathbf{E}\left[-\int_{0}^{1} x(t)f'(t)dt - x(1)f'(1)\right],\end{aligned} \tag{6}$$

where f' denotes the derivative of f and we have used the fact that

$$\mathbf{E}[\int_{0-}^{1} m(t)dx(t)] = \mathbf{E}[\int_{0-}^{1} m(1)dx(t)] = \mathbf{E}[m(1)x(1)] \tag{7}$$

in the third equality (see Dellacherie and Meyer (1982, VI.57)).

Musielak (1983, Corollary 13.14, p.87) shows that (6) is a linear functional continuous in $\mathcal{T}_\varphi$ if $f' \in L^{\varphi^*}$. This is the sufficiency part.

Now consider the necessity part. Let $\psi : \mathcal{E}^\varphi$ be a $\mathcal{T}_\varphi$-continuous linear functional. By the Hahn-Banach theorem, ψ can be extended to be a continuous linear functional on the whole of E^φ, which is the space of optional processes that satisfy (1). Let Ψ denote this extension. We first show that $\Psi : E^\varphi \to \Re$ can be represented in the form:

$$\Psi(x) = \mathbf{E}\left[\int_0^1 x(t)y(t)dt + x(1)y(1)\right] \quad \forall x \in E^\varphi \tag{8}$$

for some $y \in L^{\varphi^*}$. For this, we first consider the notion of *modular convergence*: A sequence $\{x_n\} \in L^\varphi$ is said to converge in modular to x, if

$$\mathbf{E}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\, \lambda^*(dt)\right] \to 0 \quad \text{for some } \gamma > 0.$$

The sequence $\{x_n\} \in L^\varphi$ converges in norm to x, if and only if

$$\mathbf{E}\left[\int_0^1 \varphi\Big(\gamma|x_n(t) - x(t)|\Big)\, \lambda^*(dt)\right] \to 0 \quad \text{for all } \gamma > 0;$$

see Musielak (1983, Theorem 1.6, p.3). Norm convergence of x_n clearly implies modular convergence.

Musielak (1983, Theorem 13.17, p.88) shows that (8) is true under the additional restriction on φ that

$$\forall x_0 > 0 \quad \text{there exists} \quad c > 0 \quad \text{such that} \quad \frac{\varphi(\omega, x)}{x} \geq c \quad \text{for} \quad x \geq x_0 \quad \forall \omega \in \Omega.$$

Musielak (1983, Theorem 13.15, p.87), however, shows that this restriction is required to guarantee that a norm continuous linear functional on L^φ is also modular continuous. In our formulation, we do not need this restriction, since we do not deal with the generalized Orlicz space L^φ. We only consider the subspace E^φ, and on this subspace norm convergence and modular convergence are equivalent; Musielak (1983, 5.2.B, p.18). We therefore conclude that restricting our attention to E^φ allows us to relax the above described condition. The interested reader can easily verify this by consulting Musielak (1983, Theorems 13.15 and 13.17).

Now define

$$f(\omega, t) \equiv -\int_0^t y(\omega, s)ds \quad \forall t \in [0,1] \omega \in \Omega,$$

which is clearly adapted and path-wise absolutely continuous. Denoting the derivative of $f(\omega, t)$ with respect to t by $f'(\omega, t)$, it is clear that $f'(t) = -y(t)$. Reversing the arguments in deriving (6), we prove the necessity part. ▮

PROOF OF PROPOSITION 6: Let φ be integrably asymptotically linear. We will first show that if x is an element of $\mathcal{E}^1$, then it is also an element of $\mathcal{E}^\varphi$. Note the following: for all $\gamma > 0$,

$$\begin{aligned} & \mathbf{E}[\int_0^1 \varphi(\gamma|x(t)|)\chi_{\{\gamma|x(t)|\leq K\}}\, \lambda^*(dt)] \\ + \; & \mathbf{E}[\int_0^1 \varphi(\gamma|x(t)|)\chi_{\{\gamma|x(t)|\geq K\}}\, \lambda^*(dt)] \\ \leq \; & 2\mathbf{E}\left[\varphi(\gamma K)\right] + \alpha\gamma\mathbf{E}\left[\int_0^1 |x(t)|\lambda^*(dt)\right] + 2\epsilon < \infty. \end{aligned}$$

Hence x is an element of $\mathcal{E}^\varphi$.

Since $\mathcal{T}$ is weaker than $\mathcal{T}_\varphi$, we only need to show that convergence in $\mathcal{T}$ implies convergence in $\mathcal{T}_\varphi$. Consider a sequence of elements $\{x_n\}$ that converges to x in $\mathcal{T}$. This implies that $\{x_n\}$ converges in the product measure generated by P and Lebesgue measure to x. It suffices to show that the following goes to zero as $n \to \infty$ for any $\gamma > 0$:

$$\begin{aligned} & \mathbf{E}[\int_0^1 \varphi(\gamma|x_n(t) - x(t)|)\chi_{\{\gamma|x_n(t)-x(t)|\leq K\}}\, \lambda^*(dt)] \\ + \; & \mathbf{E}[\int_0^1 \varphi(\gamma|x_n(t) - x(t)|)\chi_{\{\gamma|x_n(t)-x(t)|\geq K\}}\, \lambda^*(dt)] \\ \leq \; & \mathbf{E}[\int_0^1 \varphi(\gamma|x_n(t) - x(t)|)\chi_{\{\gamma|x_n(t)-x(t)|\leq K\}}\, \lambda^*(dt)] \\ + \; & \alpha\gamma\mathbf{E}\left[\int_0^1 |x_n(t) - x(t)|\lambda^*(dt)\right] + 2\epsilon. \end{aligned}$$

The first term on the right-hand side of the inequality converges to zero by Lebesgue convergence theorem. The second term goes to zero as $n \to \infty$ by hypothesis. Since ϵ is arbitrarily small, $\|x_n - x\|_\varphi \to 0$ as $n \to \infty$. ▮

PROOF OF PROPOSITION 7: From Propositions 4, 5, and 6, we know that $g(t) = f(t) + m(t)$, where f is an absolutely continuous process and $m(t)$ is a martingale. The assertion then follows from the fact that a martingale adapted to a Brownian motion filtration can always be represented as an Itô integral (see Clark (1970, theorems 3 and 4)). ▮

PROOF OF PROPOSITION 8: Consider a sequence of sure consumption patterns that converge in $\mathcal{T}_\varphi$. The assertion then follows from Proposition 6 of Hindy, Huang, and Kreps (1991). ▮

PROOF OF PROPOSITION 9: We omit the proof as it is similar to that for Proposition 7 of Hindy, Huang, and Kreps (1991). ▮

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